Universidade de Lisboa - ISEG

Mathematical Economics

SECOND EXAM

February 4, 2021

PART I

(1) Consider the following subsets of $\mathbb{R}^2$,

$$A = [0,1] \times [0,2],$$
$$B = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\},$$
$$C = \{(x,y) \in \mathbb{R}^2 \colon x + y < 0\},$$

where the symbol $\times$ stands for "Cartesian product".

(a) State the separating hyperplane theorem.

Solution: If A and B are disjoint and convex subsets of $\mathbb{R}^n$, then A and B are separated by a hyperplane.

(b) Find a hyperplane that separates A and $D = B \cap C$.

Solution: $H((1,1),0) = \{(x,y) \in \mathbb{R}^2 \colon x + y = 0\}$.

(2) Consider the function $f : [1,+\infty[\to \mathbb{R}$ defined by

$$f(x) = \sqrt{x} + \frac{x}{4}$$

(a) Verify that f satisfies the hypothesis of the Banach fixed point theorem.

Solution:

- $[1,+\infty[$ is a closed set.
- Since $|f'(x)| = |\frac{1}{2\sqrt{x}} + \frac{1}{4}| \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, we conclude that f is a Lipschitz contraction with $\lambda = \frac{3}{4}$.
- For any $x \geq 1$ we have $f(x) = \sqrt{x} + \frac{x}{4} \geq 1 + \frac{1}{4} \geq 1$, hence $f([1,+\infty[) \subset [1,+\infty[$.

All hypothesis of the Banach fixed point are satisfied.

(b) Find the fixed point of f.

Solution: The fixed point equation $f(x) = x$ is $\sqrt{x} + \frac{x}{4} = x$ which has solutions $x = 0$ and $x = \frac{16}{9}$. Since $x \geq 1$, the fixed point is $\frac{16}{9}$.

PART II

(1) Consider the function

$$f(x, y, z) = x^2 - 3xy + 4y^2 + z^2 \,.$$

Decide if f is strictly convex or strictly concave.

Solution:

$$D^2 f(x, y, z) = \begin{bmatrix} 2 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The leading principal minors are $\Delta_1 = 2$, $\Delta_2 = 7$ and $\Delta_3 = 14$. Since are all positive, $D^2 f(x, y, z) > 0$ for every $(x, y, z) \in \mathbb{R}^3$ and f is strictly convex.

(2) Solve the following problem:

$$\text{minimize } x^2 - 2x + y^2 + z^2$$
$$\text{subject to } 2x + y^2 = 2z$$

Explain carefully all the steps in your reasoning.

Solution:

- The Lagrangian

$$L(x, y, z, \lambda) = x^2 - 2x + y^2 + z^2 + \lambda(2x + y^2 - 2z)$$

- The critical points of L satisfy

$$\begin{cases} 2x - 2 + 2\lambda = 0 \\ 2y + 2y\lambda = 0 \\ 2z - 2\lambda = 0 \\ 2x + y^2 = 2z \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = 0 \\ z = \frac{1}{2} \\ \lambda = \frac{1}{2} \end{cases}$$

- Since $L(x, y, z, \frac{1}{2})$ is a convex function, the point $(\frac{1}{2}, 0, \frac{1}{2})$ is the solution to the minimization problem.

PART III

(1) Solve the initial value problem

$$tx' = x^2, \quad x(1) = 1.$$

Solution:

- $F(z) = \int_1^z \frac{1}{u^2}\, du = 1 - \frac{1}{z}$
- $G(t) = \int_1^t \frac{1}{s}\, ds = \log t$
- Solving $F(x(t)) = G(t)$ we get $1 - \frac{1}{x(t)} = \log t$ which gives $x(t) = \frac{1}{1-\log t}$.

(2) Consider the matrix

$$A = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}$$

(a) Find the Jordan normal form J of A.

Solution: Jordan of type II:

$$J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

(b) Compute the exponential matrix e^{tA}.

Solution:
The matrix P is

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

and the exponential matrix of type II is

$$e^{tJ} = e^{-t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Thus

$$e^{tA} = Pe^{tJ}P^{-1} = e^{-t} \begin{bmatrix} 1-2t & 4t \\ -t & 1+2t \end{bmatrix}$$

PART IV

(1) Consider the following optimal control problem in which x is the state variable and u is the control variable

$$\max_{u(\cdot)} \int_0^T \left(\frac{1}{2}\,u(t)^2 - x(t)\right) dt$$

$$\text{subject to}$$

$$\dot{x} = x - u, \text{for}\, t \in [0, T)$$

$$x(0) = 1$$

where $T > 0$ is finite and given.

(a) Write the optimality conditions according to the Pontryiagin's maximum principle. (1 point)

(b) Find the explicit solution to the problem. In particular, find the explicit solution for the terminal time T. (1.5 points)

(2) Consider a version of the benchmark consumption-investment problem, assuming that $0 < \beta < 1$)

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$\text{subject to}$$

$$a_{t+1} = \frac{a_t}{\beta} - c_t, \text{ for } t \in \{0, \dots \infty\}$$

$$a_0 \,\text{given}$$

$$\lim_{t \to \infty} \beta^{-t} a_t \geq 0$$

where a_t is the net asset position at the beginning of period t, and c_t is consumption in period t.

(a) Write the Hamilton-Jacobi-Bellman equation. Find the optimality condition for consumption. (1 point)

(b) Find the solution to the problem, $\left\{a_t^*, c_t^*\right\}_{t=0}^{\infty}$ (hint: consider the trial function $V(a) = x_0 + x_1 \ln(a)$ in which x_0 and x_1 are undetermined constants). (1.5 points)

Economia Matemática: ME MEMF MMF 2020-2021
Part:IV
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18.12.2020

Solutions
Question 1:

$$\max_{u(\cdot)} \int_0^T \left(\frac{1}{2}\, u(t)^2 - x(t)\right) dt$$
$$\text{subject to}$$
$$\dot{x} \;= x - u, \text{for } \; t \in [0, T)$$
$$x(0) = 1$$

(a) Hamiltonian function $H(u, x, \psi) = \frac{u^2}{2} - x + \lambda(x - u)$. The first order conditions are

$$\frac{\partial H(t)}{\partial u(t)} = 0 \iff u(t) = \lambda(t), \text{ for } \; t \in [0, T]$$
$$\dot{\lambda} \; = -\frac{\partial H(t)}{\partial x(t)} = 1 - \lambda(t), \text{ for } \; t \in [0, T]$$
$$\lambda(T) = 0, \text{ for } \; t = T$$
$$\dot{x} = x(t) - u(t), \text{ for } \; t \in [0, T]$$
$$x(0) = 1, \text{ for } \; t = 0$$

(b) Solving the Euler equation yields $\lambda(t) = 1 + (\lambda(0) - 1)\, e^{-t}$, where $\lambda(0)$ is unknwon. Using the transversality condition, we find $\lambda(0) = 1 - e^T$. Then $\lambda(t) = 1 - e^{T-t}$. As $u(t) = \lambda(t)$, the budget constraint becomes $\dot{x} \; = x - 1 + e^{T-t}$. The solution to this equation

$$x(t) = e^t \left(x(0) - \int_0^t e^{-s} \left(1 - e^{T-s}\right) ds\right)$$
$$= e^t \left(1 + e^{-s}\Big|_{s=0}^{t} - \frac{1}{2}\, e^{T-2s}\Big|_{s=0}^{t}\right)$$

using the initial condition. Then the solution is

$$x^*(t) = 1 - \frac{1}{2} e^T \left(e^{-t} - e^t\right), \, t \in [0, T]$$
$$u^*(t) = 1 - e^{T-t}, \, t \in [0, T]$$

Therefore $x^*(T) = \frac{1}{2}\left(1 + e^{2T}\right)$ and $u^*(T) = 0$.

Question 2: The problem

$$\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$\text{subject to}$$

$$a_{t+1} = \frac{a_t}{\beta} - c_t, \text{ for } t \in \{0, \dots \infty\}$$

$$a_0 \text{ given}$$

$$\lim_{t\to\infty} \beta^{-t} a_t \geq 0$$

(a) The HJB equation is $V(a) = \max_c \{ \ln(c) + \beta V(A)\}$ where $A = A(a,c) \equiv \dfrac{a}{\beta} - c$. The optimality condition for consumption is $\beta c^* V'(A) = 1$.

(b) Let $c^* = C(a)$ be the solution to this (implicit) equation. Substituting in the HJB equation yields the optimum HJB equation $V(a) = \ln(C(a)) + \beta V(\tilde{A}(a))$ where $\tilde{A}(a) = A(a, C(a))$.

Conjecture the trial function $V(a) = x_0 + x_1 \ln(a)$, where x_0 and x_1 are undetermined coefficients. Then we find

$$C(a) = \frac{a}{\beta(1+\beta x_1)}, \text{ and } \tilde{A}(a) = \frac{x_1 a}{1+\beta x_1}.$$

Substituting in the optimum HJB equation yields

$$(1-\beta)x_0 + \ln(\beta(1+\beta x_1)) - \beta x_1 \ln\left(\frac{x_1}{1+\beta x_1}\right) = \ln(a)(1-(1-\beta)x_1)$$

Equating both sides to zero, allows us to find that our conjecture was right, and, furthermore,

$$x_1 = \frac{1}{1-\beta} \quad x_0 = -\frac{1}{1-\beta} \ln\left(\frac{\beta}{1-\beta}\right)$$

Therefore, substituting in function $C(a)$ we find the policy function

$$c^* = \left(\frac{1-\beta}{\beta}\right) a.$$

Substituting in the constraint and taking the initial condition we have

$$a^*_{t+1} = \frac{a^*_t}{\beta} - \left(\frac{1-\beta}{\beta}\right) a^*_t = a^*_t, \text{ for } t \in \{0, \dots, \infty\}$$

$$a^*_0 = a_0$$

Therefore the solution for the problem is stationary in time,

$$a^*_t = a_0, \text{ and } c^*_t = \left(\frac{1-\beta}{\beta}\right) a_0, \text{ for every } t \in \{0, \dots, \infty\}$$